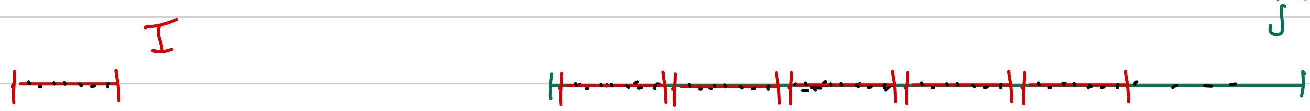
# Math 564: Advance Analysis 1
# Lecture 8

Recall that $\mathbb{E}_{\mathbb{Q}}$ is the coset eq. rel. of $\mathbb{Q} \leq \mathbb{R}$. It is also the orbit eq. rel. of the translation action $\mathbb{Q} \curvearrowright \mathbb{R}$: $x \,\mathbb{E}_{\mathbb{Q}}\, y \;:\Leftrightarrow\; x-y \in \mathbb{Q} \Leftrightarrow x = y+q$ for some $q \in \mathbb{Q}$.

**Theorem.** $\mathbb{E}_{\mathbb{Q}}$ is $\lambda$-ergodic.

**Proof.** Suppose towards a contradiction that there is an $\mathbb{E}_{\mathbb{Q}}$-invariant meas. set $A$ such that both $A$ and $A^c$ are positive measure.
Note that invariance means that $q + A = A$ for all $q \in \mathbb{Q}$.
By the 99%, let $J$ be a bdd interval with $\text{lh}(J) > 0$ whose 99% is $A^c$.
By the "arbitrarily small" 99%, there is an interval $I$ whose 99% is $A$ and s.t. $0 < \text{lh}(I) < 1\%$ of $\text{lh}(J) =: \varepsilon$.




Note that for any $q \in \mathbb{Q}$, $q+I$ is 99% $q + A = A$, because Lebesgue measure is translation invariant.
It is enough to cover 98% of $J$ by pairwise disjoint rational translates of $I$, because then $J$ would be $0.98 \cdot 99\% > 96\%$ $A$, contradicting that $J$ is 99% $A^c$.
To this end, let $n := \lceil \text{lh}(J)/\text{lh}(I) \rceil$ and take disjoint rational translates of $I$: $q_0 + I < q_1 + I < \dots < q_k + I$ so that each gap is $< \varepsilon/n$, except possibly the gap between $q_k + I$ and the right endpoint of $J$, which is $\leq 1\%$ of $\text{lh}(J)$.
This is possible by the density of $\mathbb{Q}$ in $\mathbb{R}$. Then we've indeed covered

all $J$ except for $\frac{\varepsilon}{n}\cdot(k+1)+\varepsilon \leq \frac{\varepsilon}{n}\cdot n+\varepsilon = 2\varepsilon$ much measure. But $2\varepsilon$ is $2\%$ of $J$, so we've covered $98\%$ of $J$ with disjoint rational translates of $I$. $\square$

For an eq. rel. $E$ on $X$, the **$E$-saturation** of a set $A \subseteq X$ is the set $[A]_E := \bigcup_{a\in A} [a]_E = \{x \in X : \exists a \in A \text{ with } a E x\}$.

For $\mathbb{E}_{\mathbb{Q}}$, this has a nicer form: for $A \subseteq \mathbb{R}$, $[A]_{\mathbb{E}_{\mathbb{Q}}} = \bigcup_{q\in\mathbb{Q}} (q+A)$.
So if $A$ is $\lambda$-measurable, so is $[A]_{\mathbb{E}_{\mathbb{Q}}}$.

**Corollary.** $\mathbb{E}_{\mathbb{Q}}|_A := \{(x,y) \in A^2 : x \mathbb{E}_{\mathbb{Q}} y\}$ is $\lambda$-ergodic for each $\lambda$-measurable non-null set $A \subseteq \mathbb{R}$.

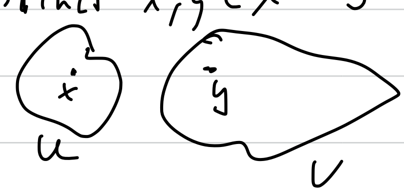
**Proof.** If $A = B \sqcup C$, where both $B, C$ are $\mathbb{E}_{\mathbb{Q}}|_A$-invariant and measurable non-null, then $[B]_{\mathbb{E}_{\mathbb{Q}}}$ and $[C]_{\mathbb{E}_{\mathbb{Q}}}$ are still disjoint, and also measurable and non-null, and $\mathbb{E}_{\mathbb{Q}}$-invariant, contradicting the ergodicity of $\mathbb{E}_{\mathbb{Q}}$. $\square$

We'll show in the next HW that every transversal of an ergodic eq. rel. is non-measurable.

## Haar Measures.

A **topological group** is a group $G$ equipped with a topology that makes multiplication $\cdot : G^2 \to G$ and the inverse function $()^{-1} : G \to G$ continuous.

Also, a top space $X$ is called
(i) **Hausdorff** if for any distinct $x, y \in X$ $\exists$ disjoint open $U, V \subseteq X$ s.t. $x \in U$ and $y \in V$



(ii) **locally compact** if every pt $x \in X$ admits a compact neighbourhood $K$ (i.e. $x \in \text{int}(K)$ and $K$ is compact).

**Examples.**
- Ctbl discrete groups
- $\mathbb{R}^d$, $d < \infty$
- $2^{\mathbb{N}} \cong (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ which is a gp under coordinatewise binary addition.
- $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$ with $\cdot$
- $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ with $\cdot$
- $GL_n(\mathbb{R}) :=$ the gp of invertible real matrices.
- $S^1 \le \mathbb{C}$ is a compact Hausdorff gp. $\bigcirc \cong \mathbb{R}/\mathbb{Z}$.

**Haar's theorem.** Every locally compact Hausdorff top. group admits a unique (up to scaling) left (or right) translation-invariant Borel measure that is positive on nonempty opens and finite on compacts. (Being positive on open sets is automatic if $G$ is $2^{nd}$ ctbl.) In particular, if $G$ is compact then it admits a unique left (or right) translation invariant prob measure.
This measure is call the/a **Haar measure**.

Thus, the Lebesgue measure is a Haar measure on $\mathbb{R}^d$.
The Bernoulli$(\frac{1}{2})$ is the Haar measure on $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$.
The counting measure is a Haar measure on ctbl discrete groups.

**Borel measures on $\mathbb{R}$.** We know that Lebesgue measure on $\mathbb{R}$ is the unique (up to scaling) translation-invariant measure on $\mathbb{R}$ among

all Borel measures on $\mathbb{R}$ that are finite on bdd sets.
We'd like to understand all Borel measures on $\mathbb{R}$ that are finite on bdd sets, e.g. $\delta_0$ or $\frac{1}{3}\delta_0 + \frac{2}{3}\delta_1$.

Let $\mu$ be such a measure on $\mathbb{R}$. Consider $f_\mu : \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto \begin{cases} \mu((0,x]) & \text{if } x \geq 0 \\ -\mu((x,0]) & \text{if } x < 0 \end{cases}$, so $f_\mu(0) = \mu((0,0]) = \mu(\emptyset) = 0$.

Note that

(i) $f_\mu$ is increasing (non-strictly) by the monotonicity of $\mu$.

(ii) $f_\mu$ is right-continuous, i.e. whenever $x_n \searrow_{n\to\infty} x$, $f_\mu(x_n) \searrow_{n\to\infty} f(x)$.

Proof. Let $x \geq 0$ and let $x_n \searrow x$.
Then $f_\mu(x) = \mu((0,x]) = \lim_n \mu((0,x_n]) = \lim_n f_\mu(x_n)$
because $(0,x] = \bigcap_n^{\downarrow} (0,x_n]$ and $\mu((0,x_0]) < \infty$.

Let $x < 0$ and let $x_n \searrow x$. Then $f_\mu(x) = -\mu((x,0])$
$= \lim_n -\mu((x_n,0]) = \lim_n f_\mu(x_n)$ because

$$(x,0] = \bigcup_n^{\uparrow} (x_n,0]. \qquad \square$$

(iii) $\mu((a,b]) = f_\mu(b) - f_\mu(a)$.

Remark. When $\mu$ is finite, the function $F_\mu : \mathbb{R} \to \mathbb{R}$
$x \mapsto \mu((-\infty,x])$
also satisfies (i) - (iii) and is called in probability the distribution of $\mu$.

What we have shown is the first part of the following

**Theorem.** (a) For each Borel measure $\mu$ finite on bdd sets there is a unique (up to a constant) function $f_\mu : \mathbb{R} \to \mathbb{R}$ with $\mu((a,b]) = f_\mu(b) - f_\mu(a)$, $\forall a<b \in \mathbb{R}$. Such a function is automatically increasing and right-continuous. It is also bdd if $\mu < \infty$.

(b) Conversely, for any increasing right-continuous $f : \mathbb{R} \to \mathbb{R}$, there is a unique Borel measure $\mu_f$ with $\mu_f((a,b]) = f(b) - f(a)$. (In particular, $f_{\mu_f} - f$ is constant.)

**Proof.** (a) We already proved it except of uniqueness, but if $f$ and $g$ are two such functions, then for any $x \geq 0$, we have $f(x) - f(0) = \mu((0,x]) = g(x) - g(0)$, so $f(x) - g(x) = f(0) - g(0)$ $\forall x \geq 0$. Similarly for $x < 0$, so $f - g$ is constant.

(b) Let $\mathcal{A}$ be the algebra generated by these half-intervals $(a,b]$. Then each $A \in \mathcal{A}$ is a finite disjoint union of (potentially unbdd) half-intervals. Define $f(\pm\infty) := \lim_{x \to \pm\infty} f(x)$.
Then we define $\mu_f$ on $\mathcal{A}$ by declaring $\mu_f((a,b]) := f(b) - f(a)$. By the same proof as for Lebesgue measure, this is a well-defined finitely additive measure on $\mathcal{A}$. In particular, it is ctbly-superadditive.
We show that it is also ctbly-subadditive. We run the same compactness argument as for Lebesgue measure. Again, the main point to show is that if $(a,b] = \bigsqcup_n (a_n, b_n]$, where $a, b \in \mathbb{R}$, then

$$\mu_f((a,b]) \leq \sum_n \mu_f((a_n, b_n]).$$